Graph Theory

UNIT - I

1.1 Graph Theory Fundamental

A graph is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself. The concept of graphs in graph theory stands up on some basic terms such as point, line, vertex, edge, degree of vertices, properties of graphs, etc.

- **Point**

A point is a particular position in a one-dimensional, two-dimensional, or three-dimensional space. For better understanding, a point can be denoted by an alphabet. It can be represented with a dot.

Example: ● a

Here, the dot is a point named ‘a’.

- **Line**

A **Line** is a connection between two points. It can be represented with a solid line.
Example: a● ————————● b

Here, ‘a’ and ‘b’ are the points. The link between these two points is called a line.

- **Vertex**

A vertex is a point where multiple lines meet. It is also called a **node**. Similar to points, a vertex is also denoted by an alphabet.

Example: ● a
Here, the vertex is named with an alphabet ‘a’.

- **Edge**

An edge is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.

Example: a● ————————● b

Here, ‘a’ and ‘b’ are the two vertices and the link between them is called an edge.

1.2 What is Graph?

Before we define the definition of Graph, let’s understand what is graph. Computers are often linked with one another so that they can interchange information. We want to send message from computer A to computer B. There are number of intermediate computers i.e. C, D, and E. Most important question is “How can we send a message from computer A to computer B using the fewest possible intermediate computers?” Lets us see few examples of interconnected computers.

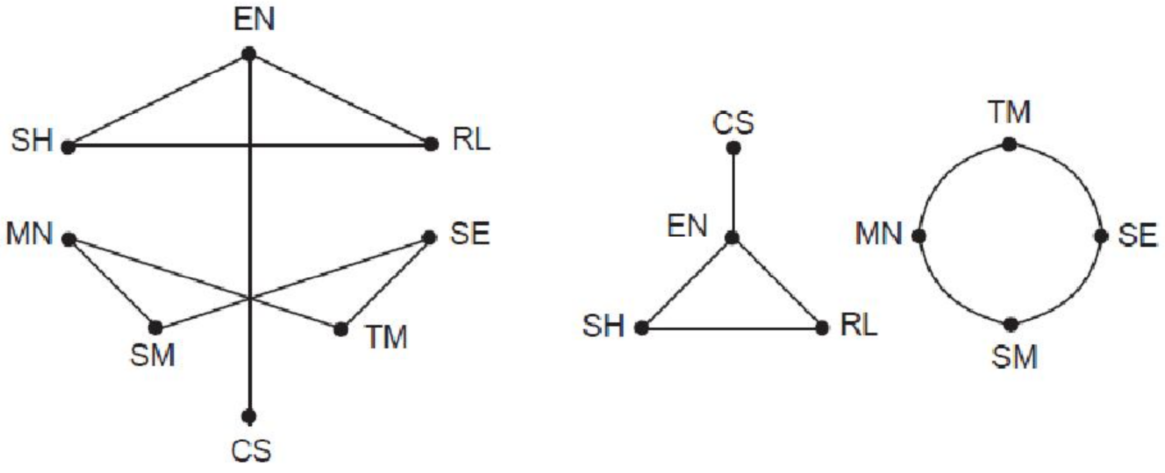


Fig 1.1

1.3 Basic Concepts in Graph Theory

Computers are connected by dots with label called vertices. The connections are indicated by lines called edges. When lines cross, they should be thought of as cables that lie on top of each other — not as cables that are joined.

Definition: A simple graph G is a pair G = (V, E) where
V is a finite set, called the vertices of G, and
E is called the edges of G

In the example above, the vertices are the computers and a pair of computers is in E if and only if they are connected. Both V and E are finite set and represented by |V|=n and |E|=m.

Example 1

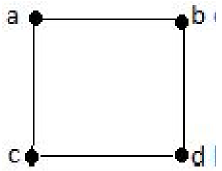


Fig 1.2

In the above example, ab, ac, cd, and bd are the edges of the graph. Similarly, a, b, c, and d are the vertices of the graph.
V = {a, b, c, d}
E={ab, ac, cd, bd}

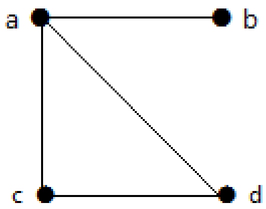


Fig 1.3

In this graph, there are four vertices a, b, c, and d, and four edges ab, ac, ad, and cd.
V = {a, b, c, d}
E = {ab, ac, ad, cd}

Example 2 (Routes between cities):
Imagine four cities named A,B,C and D. Between these cities there are various routes of travel, denoted by a, b, c, d, e, f and g. Here is picture of this situation:

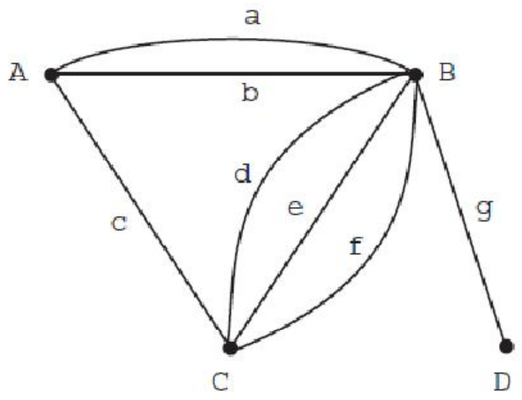


Fig 1.4

Looking at this picture, we see that there are three routes between cities B and C. These routes are named d, e and f. This picture give us only information about the interconnections between cities. Not other information like the nature of these routes (rough road, freeway, rail, etc. It also not defined the geographical distance between two cities. The object shown in this picture is called a graph.

It is possible that there can be more than one route connecting a pair of cities; e.g., d, e and f connecting cities B and C in the figure. Let's is redefine the definition of Graph to deal with such situations.

Definition (Graph) A graph is a triple G = (V,E, ϕ) where

- V is a finite set, called the vertices of G,
- E is a finite set, called the edges of G, and
- ϕ is a function with domain E and codomain P2(V).

In the pictorial representation of the cities graph, G = (V,E, ϕ) where

V = {A,B,C,D}, E = {a, b, c, d, e, f, g}

and

$$\phi = \begin{pmatrix} a & b & c & d & e & f & g \\ \{A,B\} & \{A,B\} & \{A,C\} & \{B,C\} & \{B,C\} & \{B,C\} & \{B,D\} \end{pmatrix}$$

This definition of graph tells us that to specify a graph G it is necessary to specify the sets V and E and the function ϕ. The set V is represented clearly by dots (•) i.e. the city name. The set E is the route between two cities. The function ϕ is determined by comparing the name attached to a route with the two cities connected by that route. Thus, the route name d is attached to the route with endpoints B and C. This means that ϕ (d) = {B,C}.

The function ϕ is sometimes called the incidence function of the graph. The two elements of ϕ (x) = {u, v}, for any x ∈ E, are called the vertices of the edge x, and we say u and v are joined by x. We also say that u and v are adjacent vertices and that u is adjacent to v or, equivalently, v is adjacent to u. For any v ∈ V , if v is a vertex of an edge x then we say x is incident on v. Likewise, we say v is a member of x, v is on x, or v is in x. Of course, v is a member of x actually means v is a member of ϕ (x).

Here are two additional pictures of the same cities graph given above:

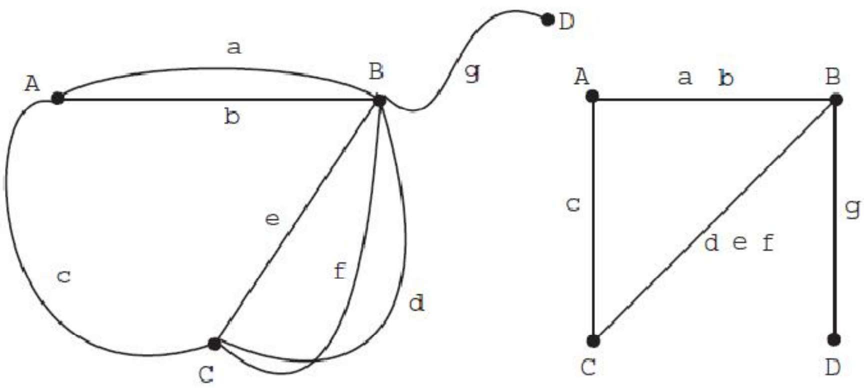


Fig 1.5

The graphs look very different but exactly the same set V and function ϕ are specified in each case.

1.4 Loops in Graph

In a graph, if an edge is drawn from vertex to itself, it is called a loop.

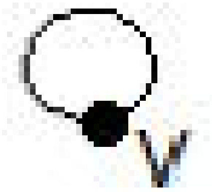


Fig 1.6

In the above graph, V is a vertex for which it has an edge (V, V) forming a loop.

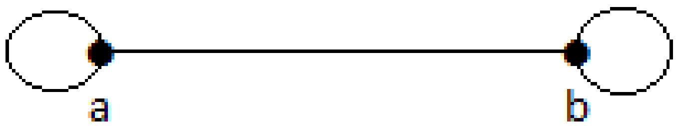


Fig 1.7

In this graph, there are two loops which are formed at vertex a, and vertex b.

1.5 Degree of Vertex

Let G = (V, E) be a graph and $v \in V$ a vertex. The degree of v is the number of $e \in E$ is incident on v. The degree of v is written as deg(v) or d(v).

In a simple graph with *n* number of vertices i.e. |V| = n, the degree of any vertices v is:

$\deg(v) \leq n - 1 \ \forall v \in G$

A vertex can form an edge with all other vertices except by itself. So the degree of a vertex will be up to the **number of vertices in the graph minus 1**. This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a Simple Graph.

Degree of vertex can be considered under two cases of graphs:

- Undirected Graph
- Directed Graph

1.5.1 Degree of Vertex in an Undirected Graph

An undirected graph has no directed edges. Consider the following examples.

Example 1

Take a look at the following graph:

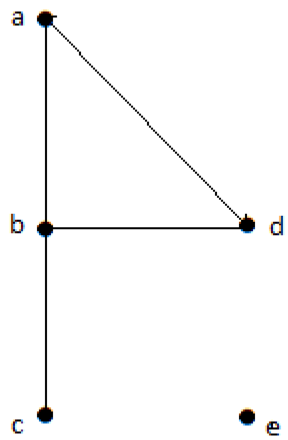


Fig 1.8

In this Undirected Graph,

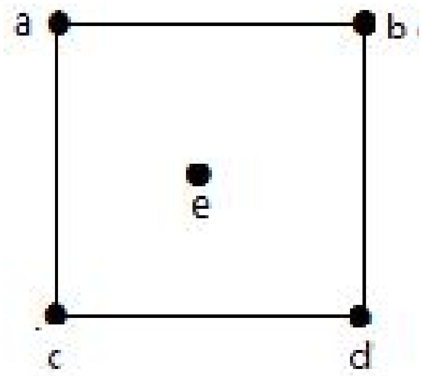
- deg(a) = 2, as there are 2 edges meeting at vertex 'a'.
- deg(b) = 3, as there are 3 edges meeting at vertex 'b'.
- deg(c) = 1, as there is 1 edge formed at vertex 'c'

  So 'c' is a **pendent vertex**.
- deg(d) = 2, as there are 2 edges meeting at vertex 'd'.
- deg(e) = 0, as there are 0 edges formed at vertex 'e'.

  So 'e' is an **isolated vertex**.

Example 2

Take a look at the following graph:



In this graph,

deg(a) = 2, deg(b) = 2, deg(c) = 2, deg(d) = 2, and deg(e) = 0.

The vertex 'e' is an isolated vertex. The graph does not have any pendent vertex.

Fig 1.8

1.5.2 Degree of Vertex in a Directed Graph

In a directed graph, each vertex has an **indegree** and an **outdegree**.

Indegree of a Graph

- Indegree of vertex v is the number of edges which are coming into the vertex v.
- **Notation:** $deg^{+}(v)$.

Outdegree of a Graph

- Outdegree of vertex V is the number of edges which are going out from the vertex V.
- **Notation:** $deg^{-}(v)$.

Consider the following examples.

Example 1
Take a look at the following directed graph. Vertex ‘a’ has two edges, ‘ad’ and ‘ab’, which are going outwards. Hence its outdegree is 2. Similarly, there is an edge ‘ga’, coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.

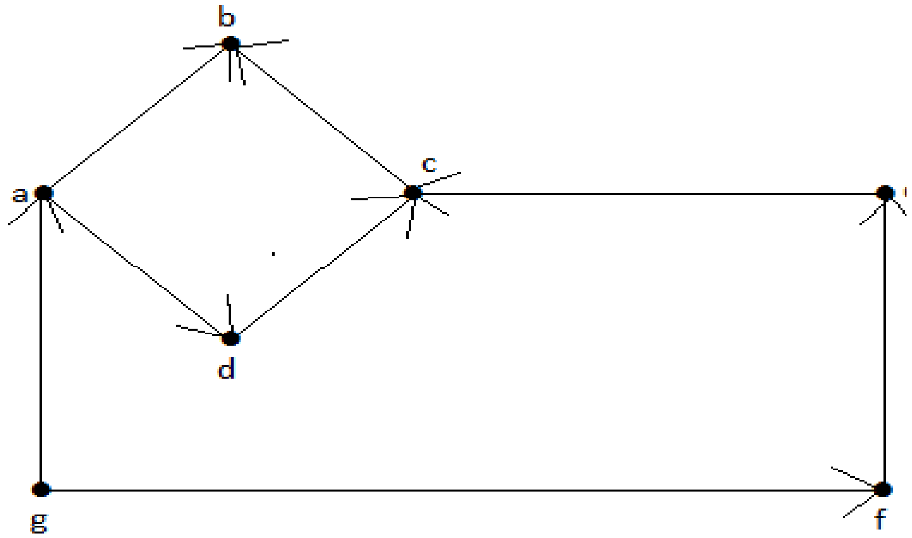


Fig 1.9

The indegree and outdegree of other vertices are shown in the following table:

Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

Example 2
Take a look at the following directed graph. Vertex ‘a’ has an edge ‘ae’ going outwards from vertex ‘a’. Hence its outdegree is 1. Similarly, the graph has an edge ‘ba’ coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.

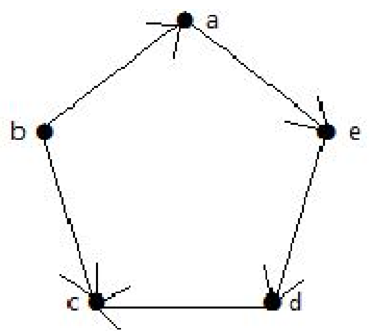


Fig 1.10

The indegree and outdegree of other vertices are shown in the following table:

Vertex	Indegree	Outdegree
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

Euler's handshaking lemma
The sum the degrees of all vertices in graph G is twice the number of edges in G. That is

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Theorem:
The number of vertices with odd degree in a graph is always even
Proof:

According to Handshaking lemma

$$\sum_{i=1}^{n} d(v_i) = 2e \text{ ---------------------------(1)}$$

Sum of degree of all vertices is always an even, so

$$\sum_{i=1}^{n} d(v_i) = \text{even --(2)}$$

If we consider the vertices with odd degree and even degree separately, the quantity in the eq (1) can be expressed as sum of vertices with odd degree and vertices of even degree.

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k). \text{ ------------------------- (3)}$$

Sum of degree of vertices with even degree | sum of degree of vertices odd degree

Since the left hand side of the eq (3) is even, and the first expression on the right hand side is even (being a sum of even numbers), the send expression must be even

$$\sum_{\text{odd}} d(v_k) = \text{an even number.}$$

Because each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum as an even number. Hence the theorem.
Examples:
Determine the number of edges in a graph having 6 vertices, 2 having a degree of 4 and 4 having a degree of 2?

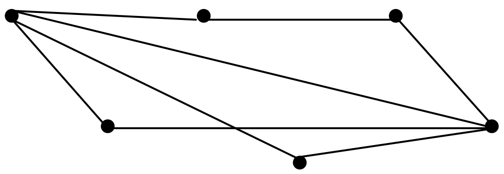
Using handshake lemma

$$\sum_{i=1}^{6} d(v_i) = 2e$$

$$= 2*4 + 4*2$$
$$= 16$$

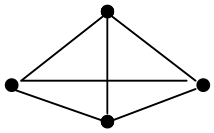
=> 2e = 16
=> e = 8
so number of edges = 8

Let's draw the graph; graph contains 6 vertices and 8 edges



Example2:
Find the maximum number of edges in a simple graph?

In a simple graph, between two of vertices only one edge is allowed. Parallel edge is not allowed as this make it multigraph.

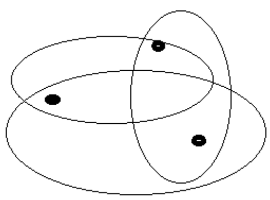


For 4 Vertices, maximum number of edges is 6

For n number of vertices, maximum number of edges would be possible when there is an edge between any two pair of vertices

Let e_n = number of combinations of pair of vertices

If we have three vertices, then maximum number of combinations of pair of vertices be



$$e_n = {}^{n}c_2 = \frac{n!}{(n-2)!\ n!}$$

$$= \frac{n\,(n-1)\,(n-2)!}{(n-2)! * 2}$$

$$= \frac{n\,(n-1)}{2}$$

e.g. n=3

en = 3(3-1)/2

= 3

n=4

en = 4 (4-1)/2

= 6

1.6 Special types of vertices

By using degree of a vertex, we have two special types of vertices.

- **Pendent Vertex**

A vertex with degree one is called a pendent vertex.

Example

a● ________________ ● b

Here, in this example, vertex ‘a’ and vertex ‘b’ have a connected edge ‘ab’. So with respect to the vertex ‘a’, there is only one edge towards vertex ‘b’ and similarly with respect to the vertex ‘b’, there is only one edge towards vertex ‘a’. Finally, vertex ‘a’ and vertex ‘b’ has degree as one which are also called as the pendent vertex.

- **Isolated Vertex**

A vertex with degree zero is called an isolated vertex.

Example

a● ● b

Here, the vertex ‘a’ and vertex ‘b’ has a no connectivity between each other and also to any other vertices. So the degree of both the vertices ‘a’ and ‘b’ are zero. These are also called as isolated vertices.

1.7 Adjacency

Here are the norms of adjacency:

- In a graph, two vertices are said to be **adjacent**, if there is an edge between the two vertices. Here, the adjacency of vertices is maintained by the single edge that is connecting those two vertices.

- In a graph, two edges are said to be adjacent, if there is a common vertex between the two edges. Here, the adjacency of edges is maintained by the single vertex that is connecting two edges.

Example 1

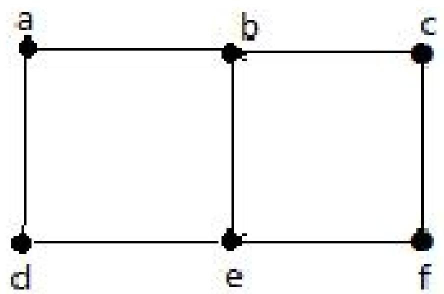
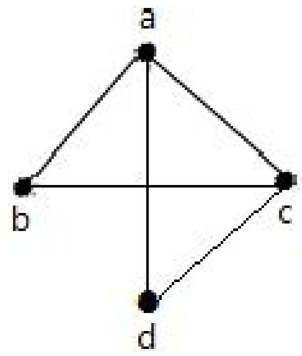


Fig 1.11

In the above graph:

- ‘a’ and ‘b’ are the adjacent vertices, as there is a common edge ‘ab’ between them.
- ‘a’ and ‘d’ are the adjacent vertices, as there is a common edge ‘ad’ between them.
- ‘ab’ and ‘be’ are the adjacent edges, as there is a common vertex ‘b’ between them.
- ‘be’ and ‘de’ are the adjacent edges, as there is a common vertex ‘e’ between them.

Example 2



In the above graph:

- 'a' and 'd' are the adjacent vertices, as there is a common edge 'ad' between them.

- 'c' and 'b' are the adjacent vertices, as there is a common edge 'cb' between them.
- 'ad' and 'cd' are the adjacent edges, as there is a common vertex 'd' between them.

- 'ac' and 'cd' are the adjacent edges, as there is a common vertex 'c' between them.

1.8 Parallel Edges

In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

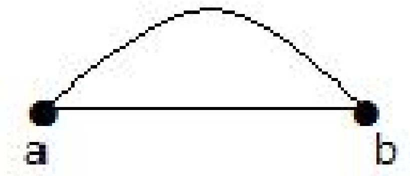


Fig 1.13

In the above graph, 'a' and 'b' are the two vertices which are connected by two edges 'ab' and 'ab' between them. So it is called as a parallel edge.

1.9 Multi Graph

A graph having parallel edges is known as a Multigraph

Example 1

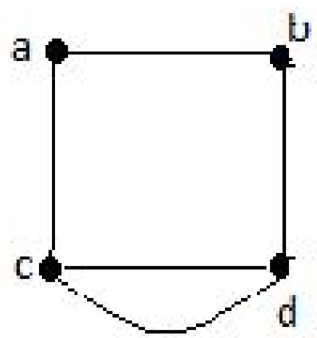


Fig 1.14

In the above graph, there are five edges 'ab', 'ac', 'cd', 'cd', and 'bd'. Since 'c' and 'd' have two parallel edges between them, it a Multigraph.

Example 2

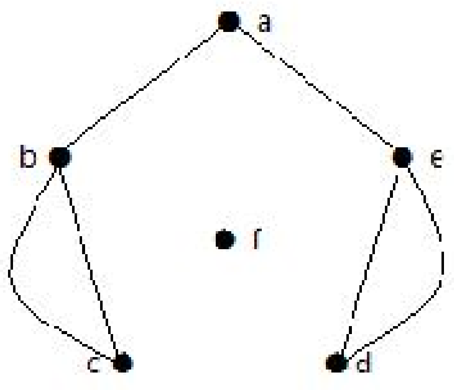


Fig 1.15

In the above graph, the vertices 'b' and 'c' have two edges. The vertices 'e' and 'd' also have two edges between them. Hence it is a Multigraph.

1.10 Degree sequence of the Graph

If the degrees of all vertices in a graph are arranged in descending or ascending order, then the sequence obtained is known as the degree sequence of the graph.

Suppose |V | = n. Let d1, d2, . . . , dn, where d1 ≤ d2 ≤ · · · ≤ dn be the sequence of degrees of the vertices of G, sorted by size. This is referred to as degree sequence of the graph G.

In the graph for routes between cities, d(A) = 3, d(B) = 6, d(C) = 4, and d(D) = 1. The degree sequence is 1,3,4,6.

Example 1

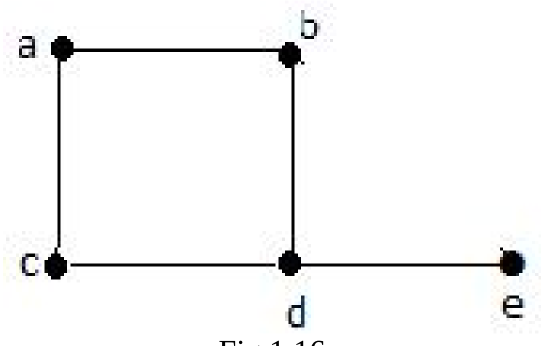


Fig 1.16

Vertex	a	b	c	d	e
Connecting To	b, c	a, d	a, d	c, b, e	d
Degree	2	2	2	3	1

In the above graph, for the vertices {d, a, b, c, e}, the degree sequence is {3, 2, 2, 2, 1}.

Example 2

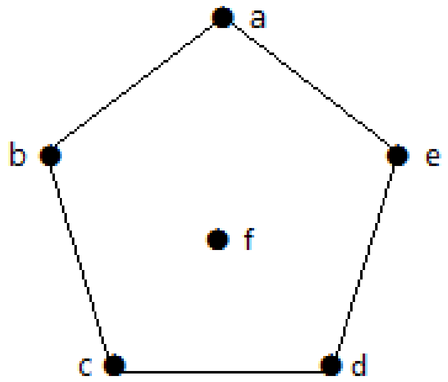


Fig 1.17

Vertex	a	b	c	d	e	f
Connecting To	b, e	a, c	b, d	c, e	a, d	-
Degree	2	2	2	2	2	0

In the above graph, for the vertices {a, b, c, d, e, f}, the degree sequence is {2, 2, 2, 2, 2, 0}.

1.11 Applications of Graph Theory

- **Electrical Engineering** – The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.

- **Computer Science** – Graph theory is used for the study of algorithms. For example,
  - Kruskal's Algorithm
  - Prim's Algorithm
  - Dijkstra's Algorithm

- **Computer Network** – The relationships among interconnected computers in the network follows the principles of graph theory.

- **Science** – The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.

- **Linguistics** – The parsing tree of a language and grammar of a language uses graphs.

- **General** – Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.